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FINITE ELEMENT APPROXIMATION OF AN OPTIMAL DESIGN PROBLEM

A. CHAKIB[1], A. NACHAOUI[2], M. NACHAOUI[1,2]

Abstract. This paper investigates shape optimization of complex thermo-fluid phenomena that occur in welding processes. The linear finite elements dicretization is accomplished. The existence of the discrete optimal solution is established. Some computational results for our approach are presented and discussed.

Keywords: Shape Optimization, Free Boundary, Non Coercive Operator, Welding, Finite Element, Genetic Algorithms.

AMS Subject Classification: 49Q10, 35R35, 65N30.

1. Introduction

In this paper, we consider a problem modeling analysis of heat transfer in a welding operation. The aim is to identify the liquid/solid interface and estimate the field temperature in the welded parts of the plate in order to predict and control the mechanical effects caused by the process on these parts (residual stresses, distortions...). The considered approach concerned only the solid part of the plate and it consists to simplify the physical phenomena occurring between the welding torch and the plate as well as the liquid bath by introducing a temperature condition imposed on the liquid/solid interface which is unknown. To solve this free boundary problem, an optimal shape design formulation was proposed in [4]. Our interest is the numerical study of the approached shape design problem, obtained by using the finite element method and the parametrization of the liquid/solid interface by Bézier curves. We are interested more precisely by showing the existence of the optimal discrete solution of this approached problem. The main difficulty of this work lies in the fact that the state problem is governed by a noncoercive operator, which complicates the study of existence. At this stage, it must be noted that in the coercive case we can show easily this result, see [8]. The proposed approach for overcoming this difficulty is based on the topological degree tools in finite dimensional spaces [6], and a uniform estimate of discrete solutions norm's. To show the efficiency of our approach, we give some numerical results.

2. Setting of the problem

We are interest by a numerical realization, using the finite element method, of the optimal shape design formulation of a welding problem given by

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$$\left\{\begin{array}{l} \text{find } \Omega^* \in \Theta_{ad} \text{ solution of} \\ J(\Omega^*) = \inf\limits_{\Omega\in\Theta_{ad}} J(\Omega) \\ \text{ where } J(\Omega) = \frac{1}{2}\int_{\Gamma_0} |T(\Omega(x,y)) - T_0|^2 \, d\sigma \\ \text{ and } T(\Omega) \text{ the solution of} \\ (SP) \left\{\begin{array}{l} K\frac{\partial T}{\partial x} = \nabla\cdot(\lambda\nabla T) + f \text{ in } \Omega \\ \lambda\frac{\partial T}{\partial \nu} = 0 \text{ on } \Gamma_0\cup\Gamma_1\cup\Gamma_2\cup\Gamma_3 \\ T = T_d \text{ on } \Gamma_4, \quad T = T_f \text{ on } \Gamma, \end{array}\right. \end{array}\right. \tag{1}$$

where the parameters in (1) are such that:
K is a constant dependent to the material characteristics (density of the plate and heat capacity,...), λ is the thermal conductivity, f is a given source term. The quantities T_d, T_0 and T_f are given temperatures. The solid part of plate Ω (see fig. 1), is defined by

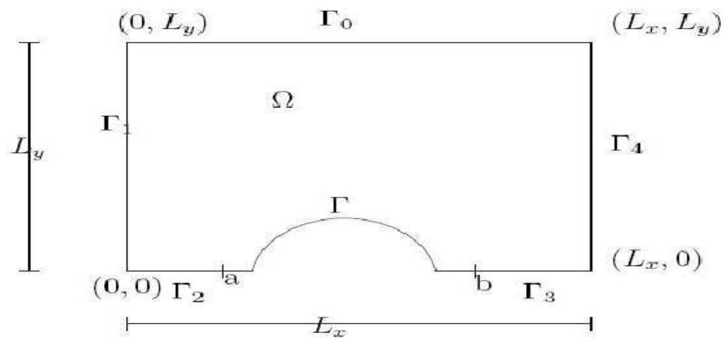

Figure 1. The solid part of the welded workpiece with interface Γ.

$$\Omega(\varphi) =]0,a[\times]0,L_y[\cup\left\{(x,y)\in I\!R^2/a\le x\le b, \varphi(x)\le y\le L_y\right\}\cup]b,L_x[\times]0,L_y[\tag{2}$$

where φ, the parametrisation of the unknowon boundary Γ, is a Lipschitz function. The set Θ_{ad} is defined by

$$\Theta_{ad} = \{\Omega(\varphi) \quad \varphi\in U_{ad}\}$$

and

$$\begin{aligned} U_{ad} \quad = \quad & \left\{ \varphi\in C([a,b]) \ / \ \exists\ a_\varphi \text{ and } b_\varphi\in[a,b] \ , \ \varphi_{|[a,a_\varphi]} = 0 \ , \ \varphi_{|[b_\varphi,b]} = 0 \text{ and } \exists\ L_0 > 0 \ / \right. \\ & \left. \left|\varphi(x)-\varphi(x')\right| \le L_0\left|x-x'\right| \ \ \forall x,x'\in[a,b] \ , \ \ 0\le\varphi(x)\le L_y \ \forall x\in[a,b] \ \right\}. \end{aligned}$$

In the sequel we suppose that the parameters of our problem are such that: $D =]0,L_x[\times]0,L_y[$,
(H_1) $\lambda\in L^\infty(D)$ and $\exists\lambda_0>0$ such that $\lambda(x)\xi\cdot\xi\ge\lambda_0|\xi|^2$ a.e $x\in D$
(H_2) $K\in L^\infty(D)$ and $f\in L^2(D)$
Let $\Gamma_D = \Gamma\cup\Gamma_4$, we define the space $H^1_{\Gamma_D}(\Omega) = \left\{u\in H^1(\Omega) \ / \ u|_{\Gamma_D} = 0 \ \right\}$ where $H^1(\Omega)$ is the Sobolev space. From the surjectivity of the trace operator from $H^1(D)$ to $H^{\frac{1}{2}}(\partial D)$, we have

$$\exists\ V \in H^1(D) \qquad \text{such that} \qquad V = \left\{\begin{array}{ll} v & \text{on} \quad]b,L_x[\times]0,L_y[\\ T_f & \text{on} \quad]0,b[\times]0,L_y[, \end{array}\right.$$

where $v\in H^1(]b,L_x[\times]0,L_y[)$ such that $v = T_d$ on Γ_4 and $v = T_f$ on $\{b\}\times[0,L_y]$.
Then a variational formulation of the state problem (SP) is the following:

$$\left\{\begin{array}{l} \text{find } u\in H^1_{\Gamma_D}(\Omega) \\ \int_\Omega \lambda\nabla u\cdot\nabla\psi \ + \int_\Omega K\,\psi\frac{\partial u}{\partial x} = \int_\Omega f\,\psi \ - \int_\Omega \lambda\nabla V\cdot\nabla\psi \ - \int_\Omega K\,\psi\frac{\partial V}{\partial x} \ \ \forall\psi\in H^1_{\Gamma_D}(\Omega). \end{array}\right. \tag{3}$$

The following existence result is proved in [4].

Theorem 1. Under assumptions (H_1) and (H_2), the problem (1) is well posed and admits at least one solution in Θ_{ad}.

3. Numerical approximation of the problem

In this section we give an approximation of (1); we shall discretize both the admissible family Θ_{ad} and the state problem (SP). We start with the first one, for this we use the piecewise spline approximations of $\Gamma(\varphi)$ locally realized by quadratic Bézier functions [8].

3.1. **Discretization of the shape optimal problem.** Let us consider a uniform partition $(a_i)_{i=0}^d$ of $[a,b]$, such that $a = a_0 < a_1 < ... < a_d = b$, $a_i = i\mu + a$, $\mu = (b-a)/d$, $i = 0,...,d$; and $a_{i+1/2}$ be the midpoint of $[a_i, a_{i+1}]$. Further let $A_i = (a_i, \varphi_i)$, $\varphi_i \in I\!R$, $i = 0,...,d$, be design nodes and $A_{i+1/2} = \frac{1}{2}(A_i + A_{i+1})$ be midpoint of the segment $\overline{A_i A_{i+1}}$, $i = 0,...,d-1$. In addition let $a_{-\frac{1}{2}} = a - \frac{\mu}{2}$, $a_{d+\frac{1}{2}} = b + \frac{\mu}{2}$, $A_{-\frac{1}{2}} = (a_{-\frac{1}{2}}, \frac{1}{2}(\varphi_0 + \varphi_1))$, $A_{d+\frac{1}{2}} = (a_{d+\frac{1}{2}}, \frac{1}{2}(\varphi_{d-1} + \varphi_d))$.

Remark 1. The triple $\{A_{i-\frac{1}{2}}, A_i, A_{i+\frac{1}{2}}\}$, is termed the control points of the Bézier function.

For a partition $(a_i)_{i=0}^d$ we associate the set $\mathcal{Q}_\mu^{ad} \subset U_{ad}$ of continuous, piecewise linear functions over $(a_i)_{i=0}^d$:

$$\mathcal{Q}_\mu^{ad} = \{\varphi_\mu \in C([a,b]) \mid \varphi_\mu|_{[a_{i-1},a_i]} \in P_1([a_{i-1}, a_i]) \ \forall i = 1,...,d\} \cap U_{ad}. \tag{4}$$

The family of admissible discretized design domains is now represented by

$$\Theta_{ad}^\mu = \{ \ \Omega(s_\mu) \quad / s_\mu \in U_{ad}^\mu, \quad \} \tag{5}$$

where

$$\begin{aligned} U_{ad}^\mu = \{ \ & s_\mu = \widetilde{s}_\mu|_{[a,b]} \in C^1([a - \tfrac{\mu}{2}, b + \tfrac{\mu}{2}]) \ / \ \widetilde{s}_\mu|_{[a_{i-\frac{1}{2}}, a_{i+\frac{1}{2}}]} \text{ is a quadratic Bézier} \\ & \text{function determined by } \{A_{i-\frac{1}{2}}, A_i, A_{i+\frac{1}{2}}\}, \\ & \text{where } A_i = (a_i, \varphi_\mu(a_i)),\ i = 0,...,d, \text{ and } \ \varphi_\mu \in \mathcal{Q}_\mu^{ad}. \ \} \end{aligned} \tag{6}$$

Now, we start the approximation of the state problem (SP). We use the finite element method with continuous piecewise linear polynomials over a triangulation of the computational domain (an appropriate approximation of $\Omega(s_\mu) \in \Theta_{ad}$). We introduce another family of regular partition $(b_i)_{i=0}^q$ of $[a,b]$, such that: $a = b_0 < b_1 < ... < b_q = b$ (not necessary equidistant), whose norm well be denoted by h. We suppose that $h \longrightarrow 0^+$ if $\mu \longrightarrow 0^+$. Let $r_h s_\mu$ be the piecewise linear Lagrange interpolate of s_μ on $(b_i)_{i=0}^q$:

$$r_h s_\mu(b_i) = s_\mu(b_i) \ \text{ and } r_h s_\mu|_{[b_{i-1},b_i]} \in P_1([b_{i-1}, b_i]) \quad \forall i = 0, \cdots, q;$$

Then the computational domain of $\Omega(s_\mu)$ is represented by $\Omega(r_h s_\mu)$. The system of all $\Omega(r_h s_\mu)$, $s_\mu \in U_{ad}^\mu$, will be denoted by $\Theta_{ad}^{\mu h}$ in what follows:

$$\Theta_{ad}^{\mu h} = \{\Omega(r_h s_\mu) \ \mid \ s_\mu \in U_{ad}^\mu\}. \tag{7}$$

Since $\Omega(r_h s_\mu)$ is already polygonal, one can construct its triangulation $\mathcal{T}(h, s_\mu)$ with the $h > 0$ and depending on $s_\mu \in U_{ad}^\mu$. We shall suppose that for $h > 0$ fixed, triangulations $\mathcal{T}(h, s_\mu)$ are topologically equivalent for all $s_\mu \in U_{ad}^\mu$. The domain $\Omega(r_h s_\mu)$ with a given triangulation $\mathcal{T}(h, s_\mu)$ will be denoted by $\Omega_h(s_\mu)$ and the approximate of Γ is noted by Γ_h. Let

$$H_h(\Omega_h(s_\mu)) = \{v_h \in C(\overline{\Omega}_h) \ \mid \ v_h|_T \in P_1(T), \ T \in \mathcal{T}(h, s_\mu)\}$$

and

$$H_{\Gamma_D}^h(\Omega_h(s_\mu)) = \{v_h \in H_h(\Omega_h(s_\mu)) \mid v_h|_{\Gamma_{D,h}} = 0\}$$

be the finite dimensional spaces associated respectively to $H^1(\Omega)$ and $H_{\Gamma_D}^1(\Omega)$. We note that the finite element method used here is the conforming one [11]. Then for any $s_\mu \in U_{ad}^\mu$, the approximation $u_h := u_h(s_\mu) \in H_{\Gamma_D}^h(\Omega_h(s_\mu))$ of $u \in H_{\Gamma_d}^1(\Omega)$ is given by: $u_h = \sum_{i=1}^N u_h(\bar{b}_i)\psi_i$, where N is the number of the nodes of $\mathcal{T}(h, s_\mu)$ lying in $\overline{\Omega}_h(s_\mu)$, $(\bar{b}_i)_{1 \le i \le N}$ is a vertex of the triangulation and $(\psi)_{i=1}^{N}$ is a basis function of $H^h_{\Gamma_D}(\Omega_h(s_\mu))$. Let $\digamma(r_h(s_\mu)) = D \setminus \Omega(r_h(s_\mu))$, we construct another family $\{\mathcal{TE}(h, s_\mu)\}$ of triangulations of $\overline{\digamma(r_h(s_\mu))}$. The union of $\mathcal{T}(h, s_\mu)$ and $\mathcal{TE}(h, s_\mu)$ define a regular triangulation of $\overline{D}$. Let V_h be a piecewise lineair Lagrange interpolant of V in $\bar{D}$.

The discrete state problem reads

$$\left\{\begin{array}{l} \text{Find } u_h \in H^h_{\Gamma_D}(\Omega_h(s_\mu)) \text{ such that } \quad \forall v_h \in H^h_{\Gamma_D}(\Omega_h(s_\mu)) \\ \displaystyle\int_{\Omega_h(s_\mu)} \lambda_h \nabla u_h \cdot \nabla v_h \ + \int_{\Omega_h(s_\mu)} K_h\, v_h \frac{\partial u_h}{\partial x} = \\ \displaystyle\int_{\Omega_h(s_\mu)} f\, v_h \ - \int_{\Omega_h(s_\mu)} \lambda_h \nabla V_h \cdot \nabla v_h - \int_{\Omega_h(s_\mu)} K_h\, v_h \frac{\partial V_h}{\partial x}, \end{array}\right. \tag{8}$$

where K_h (resp λ_h) is an approximation of K (resp λ) such that K_h (resp λ_h) is uniformly bounded, converges to K (resp λ), almost every where and satisfies the following equation:

$$\exists \lambda_0 > 0 \quad \text{independent of } h \text{ such that } \lambda_h(x)\xi \cdot \xi > \lambda_0 |\xi|^2 \quad \text{a.e } x \in D. \tag{9}$$

We approach the cost functional by the following discrete one:

$$J_h(u_h(s_\mu)) = J_h(\Omega_h(s_\mu)) = \frac{1}{2} \int_{\Gamma_0^h} |T_h(s_\mu) - T_0|^2 \, d,\sigma \tag{10}$$

where $T_h(s_\mu) = u_h(s_\mu) + V_h$ and $u_h(s_\mu) \in H^h_{\Gamma_D}(\Omega_h(s_\mu))$.

We state our discrete optimal shape problem as follows

$$\left\{\begin{array}{l} \displaystyle\inf_{s_\mu \in U^\mu_{ad}} J_h(u_h(s_\mu)), \\ \text{where } u_h(s_\mu) \text{ is solution of (8) on } \Omega_h(s_\mu), \end{array}\right. \tag{11}$$

where N is the number of the nodes of $\mathcal{T}(h, s_\mu)$ lying in $\overline{\Omega}_h(s_\mu)$. In the following we prove the existence of a solution of (11).

3.2. **Existence of the discrete optimal domain.** The basic step in the existence analysis of a solution of (11) consists in showing that solutions of (8) depend continuously on shape variations for all $h > 0$. This is based on the following lemma.

Lemma 1. $\exists\, C > 0$, $\forall\ s_\mu \in U^\mu_{ad}$ and $\forall\ \ h > 0 \qquad \|u_h(s_\mu)\|_{1,\Omega_h(s_\mu)} \leq C.$

Proof. The main difficulty of this work is to show that $\|u_h(s_\mu)\|_{1,\Omega_h(s_\mu)}$ is uniformly bounded with respect to $\Omega_h(s_\mu)$. For this we use the two following inequalities (see [1, 2, 9])

- There exists $C_0 > 0$ independent of $\Omega_h(s_\mu)$ such that $\forall u_h \in H^1_{\Gamma_D}(\Omega_h(s_\mu))$

$$C_0 \, \|u_h(s_\mu)\|_{1,\Omega_h(s_\mu)} \leq \int_{\Omega_h(s_\mu)} |\nabla u_h(s_\mu)|^2. \tag{12}$$

- There exists $C > 0$ independent of $\Omega_h(s_\mu)$ such that

$$\|u_h(s_\mu)\|_{L^4(\Omega_h(s_\mu))} \leq C|\Omega_h(s_\mu)|^{\frac{1}{4}} \, \|u_h(s_\mu)\|_{H^1(\Omega_h(s_\mu))} .$$

Then we define the set $A_k = \{(x, y) \in \Omega_h(s_\mu), |u_h(x)| > k\}$, the functions $h_k(u_h) = max(-k, min(u_h(s_\mu), k))$ and $\psi_k(u_h(s_\mu)) = u_h(s_\mu) - h_k(u_h(s_\mu))$. First we show the following uniform estimation of $\psi_k(u_h(s_\mu))$:

$(C_0 - C|A_k|^{\frac{1}{4}}) \, \|\psi_k(u_h(s_\mu))\|^2_{H^1(\Omega_h(s_\mu))} \leq | < \ell, \psi_k(u_h(s_\mu)) >_{((H^1_{\Gamma_D}(\Omega_h(s_\mu)))', H^1_{\Gamma_D}(\Omega_h(s_\mu)))} |.$

To show that the constant $(C_0 - C|A_k|^{\frac{1}{4}})$ is positive. We start by showing the uniform control of Lebesgue measure of A_k, using Tchebychev inequality and the uniform estimate of $\ln(1 + |u|)$, i.e. there exists $C_2 > 0$ independent of $\Omega_h(s_\mu)$ such that

$$|A_k| = \Big| \ \{(x, y) \in \Omega_h(s_\mu)/ \ln(1 + |w|)^2 \geq$$

$$\geq \ln(1+k)^2\} \mid \leq \frac{1}{\ln(1+k)^2} \|\ln(1+|w|)\|_{L^2(\Omega_h(s_\mu))} \leq \frac{C_2}{\ln(1+k)^2}. \tag{13}$$

Then there exists $k_0 \in \mathcal{N}^*$, such that $\forall k \geq k_0 \quad C|A_k|^{\frac{1}{4}} \leq \frac{C_0}{2}$. Taking $k = k_0$, we show that there exists $C_3 > 0$ independent of $\Omega_h(s_\mu)$ such that

$$\|\psi_{k_0}(u_h(s_\mu))\|_{H^1(\Omega_h(s_\mu))} \leq C_3.$$

Finally, using the fact that $h_{k_0}(u_h(s_\mu))u_h(s_\mu) \geq (h_{k_0}(u_h(s_\mu)))^2$, $\nabla h_{k_0}(u_h(s_\mu)) = \chi_{A_{k_0}} \nabla u_h(s_\mu)$ and inequality (12), we show the existence of $C_4 > 0$ independent of $\Omega_h(s_\mu)$ such that $\|h_{k_0}(u_h(s_\mu))\|_{H^1(\Omega_h(s_\mu))} \leq C_4$.

We can now prove the follwing theorem.

Theorem 2. Under the assumptions (9), the problem (11) admits a solution on U^μ_{ad}, for all $h > 0$ and $\mu > 0$.

Proof. for $s_\mu \in U^\mu_{ad}$ fixed and $h > 0$, we define the operator F_t, $\forall t \in [0, 1]$, by

$$\begin{aligned} F_t \quad : \quad & H^h_{\Gamma_D}(\Omega_h(s_\mu)) \to H^h_{\Gamma_D}(\Omega_h(s_\mu)), \\ & \bar{u}_h \mapsto u_h, \end{aligned}$$

where u_h is the unique solution of,

$$\int_{\Omega_h(s_\mu)} \lambda_h \nabla u_h \cdot \nabla v_h = \int_{\Omega_h(s_\mu)} f\, v_h -$$

$$-t \int_{\Omega_h(s_\mu)} K_h\, v_h \frac{\partial \bar{u}_h}{\partial x} - \int_{\Omega_h(s_\mu)} \lambda_h \nabla V_h \cdot \nabla v_h - \int_{\Omega_h(s_\mu)} K_h\, v_h \frac{\partial V_h}{\partial x}. \tag{14}$$

The a priori estimate $\|u_h\|_{1,\Omega_h(s_\mu)} < C$, with $C > 0$, allows as to build an open ball B, such that there is no fixed point of F_t on the boundary of B. Thus $deg[I - F_t, B, 0]$ is defined and independent of t, where 'deg' is the topological degree [6] and I the identity mapping in $H^h_{\Gamma_D}(\Omega_h(s_\mu))$. Since F_0 is trivial, we conclude that $1 = deg[I - F_0, B, 0] = deg[I - F_1, B, 0]$. Therefore, F_1 admits a fixed point in the interior of B which is solution of (8). For the uniqueness of the discrete solution, since the second member of (8) is linear, we show that equation (8) with second member zero, has no solution other than zero. This means that the problem is well posed. It remains to show that that solutions of (8) depend continuously on shape variations for all $h > 0$.

Let $(s^j_\mu)_j \subset U^\mu_{ad}$, we can extract a subsequence denoted again $(s^j_\mu)_j$ such that $s^j_\mu \to s^*_\mu$ in U^μ_{ad} and $\Omega_h(s^j_\mu) \to \Omega_h(s^*_\mu)$ as $j \to \infty$. According to Lemma 1, $\exists C > 0$

$$\left\|u_h(s^j_\mu)\right\|_{1,\Omega_h(s^j_\mu)} \leq C. \tag{15}$$

From Chenais's uniform extension result [3], there exist $\tilde{u}_h(s^j_\mu)$ a uniform extension of $u_h(s^j_\mu)$ from $\Omega_h(s^j_\mu)$ to D, such that

$$\exists\, M > 0 \qquad \forall j \quad \left\|\tilde{u}_h(s^j_\mu)\right\|_{1,D} < M.$$

Thus there exists a subsequence $\tilde{u}_h(s^j_\mu)$ and an element $\tilde{u}_h \in H^1(D)$,

$$\tilde{u}_h(s^j_\mu) \rightharpoonup \tilde{u}_h \quad \text{in} \quad H^1(D).$$

Let us show that $u_h = \tilde{u}_h|_{\Omega_h(s^*_\mu)}$ solves (8). It's easy to see that $u_h|_{\Gamma_4} = 0$ and using the compactness of the trace operator from $H^1(D)$ into $L^2(\Gamma)$, we show that $u_h \in H^1_{\Gamma_D}(\Omega_h(s^*_\mu))$.
It remains to show that u_h solve (8). Let $\psi_h \in H^h_{\Gamma_D}(\Omega_h(s^*_\mu))$ and $\tilde{\psi} \in H^1(D)$ be an extension of ψ_h defined by

$$\tilde{\psi} = \begin{cases} \psi_h & \text{in } \Omega_h(s^*_\mu) \\ 0 & \text{in } D \setminus \Omega_h(s^*_\mu). \end{cases}$$

Then we can construct a sequence $(\psi_n)_n$, $\psi_n \in \mathcal{D}(\bar{D})$, such that,

$$dist(supp\ \psi_n, \overline{\Gamma_D}) > 0 \ \ \forall n \in \mathcal{N} \quad \text{and} \quad \psi_n \to \tilde{\psi} \ \text{ in } H^1(D), n \to \infty.$$

Let $n \in \mathcal{N}$, since $\Omega_h(s_\mu^j) \to \Omega_h(s_\mu^*)$, there exists j_0 such that $\psi_n^h = \pi_h \psi_n|_{\Omega_h(s_\mu^j)} \in H^1_{\Gamma_D}(\Omega_h(s_\mu^j)), \forall j \geq j_0$, where $\pi_h\psi_n$ is the piecewise linear interpolation of ψ_n on $\mathcal{T}(h, s_\mu^j)$. For all $j \geq j_0$, we have

$$\begin{aligned} \int_D \chi_{\Omega_h(s_\mu^j)} \lambda_h \nabla \tilde{u}_h(s_\mu^j) \cdot \nabla \psi_n^h + \int_D \chi_{\Omega_h(s_\mu^j)} K_h \frac{\partial \tilde{u}_h(s_\mu^j)}{\partial x} \psi_n = \\ = \int_D \chi_{\Omega_h(s_\mu^j)} f \psi_n^h - \int_D \chi_{\Omega_h(s_\mu^j)} \lambda_h \nabla \pi_h V(s_\mu^j) \cdot \nabla \psi_n^h - \int_D \chi_{\Omega_h(s_\mu^j)} K_h \frac{\partial \pi_h V(s_\mu^j)}{\partial x} \psi_n. \end{aligned} \tag{16}$$

Passing to the limit first with $n \to \infty$, then with $j \to \infty$ in (16), we obtain that u_h is solution to the (8).

3.3. **Numerical algorithms.** To solve the welding problem, we developed a numerical algorithm based on a genetic algorithm procedure [10] for solving our discrete optimal shape problem (11). Genetic algorithms (GA), primarily developed by Holland [7], have been successfully applied to various optimizations problems. It is essentially a searching method based on the Darwinian principles of biological evolution. In GA a new generation of individuals is produced using the simulated genetic operations crossover and mutation. The probability of survival of generated individuals depends of their fitness: the best ones survive with the high probability, the worst die rapidly. This procedure can be summarized in the following algorithm see [10].

(1) Iteration $k = 0$, Generate randomly an admissible population.
(2) Solve (8) for each individual of population.
(3) Evaluate the fitness (10) for each individual of population.
(4) If the termination criteria is hold $J_h \leq \varepsilon$, then stop.
Else set $k = k + 1$ and go to step 5.
(5) Roulette wheel selection
(6) Applied to the selected individuals, the barycenter crossover procedure.
(7) Select randomly some individual, and applied to them the mutation.
(8) Go to step 2.

4. Numerical results

In the following, we solve the welding problem considering the workpiece D as the square $L_x = 1,\ \ L_y = 1$.

4.1. **Validation of the method against a design model.** Consider our model example (1) with the following physical data (corresponding to the aluminium variante),

$$\begin{gathered} \lambda = 0.221\ kJ.(K.m.S)^{-1}, \quad K = \rho\, C\, v_{torch}, \quad \rho = 2.37 \times 10^3\ kg.m^{-3} \\ C = 0.124\ kJ.(kg.K)^{-1}, \quad v_{torch} = -30\ mm.s^{-1}, \quad f(x,y) = 0, \\ \frac{\partial T}{\partial \nu} = 0 \quad \text{on} \quad \Gamma_0 \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \quad \text{and} \quad T_f = 659.25\ C, \quad T_d = 20\ C \end{gathered}$$

The exact boundary is taken as the Bézier curve defined by the following control point :

$$b_0 = (0.35, 0), \quad b_1 = (0.35, 15), \quad b_2 = (0.65, 0.285), \quad b_3 = (0.65, 0)$$

We solve the direct problem (SP) using finite element, the obtained solution on Γ_0 is then specified as the desired temperature T_0.

Fig. 2 illustrates the iterative convergence process as the initial guess for the free boundary moves towards the exact boundary Γ, for various numbers of iterations performed. From this figure, it can be seen that the numerically retrieved boundary is a very good approximation of the exact one.

5. Conclusions

This paper is concerned with the approximation of the welding problem formulated as a PDE optimization problem where the shape of the interface serves as the control variable. To avoid the shape differential calculus needed in a gradient like method for solving a shape optimization problem, we used a numerical algorithm based on genetic algorithm procedure, Bézier curve parametrization of the free boundary and finite element discretization of the state problem. We proved the existence of the discrete optimal solution. Our computational example confirms the efficiency of the proposed approach. The convergence can be accelerated by the parallel computation procedure.

It can be stressed that the presented method admits a straightforward generalization to three dimensions. Our future work in this class of problems will involve extensions of the present method to a time-dependent problem.

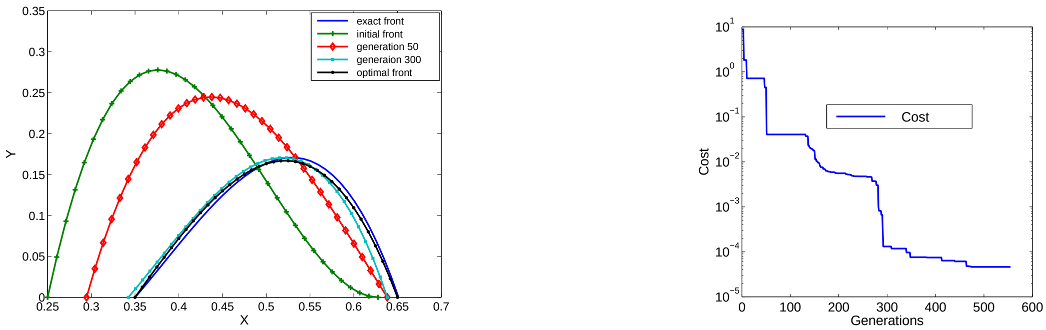

Figure 2. The cost functional decreasing. And the iterative convergence process for the unknown boundary.

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